# Options Pricing

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# Contents



# 1 Introduction

Options have been around for centuries, however they became more known and gained a lot more interest from academics in 1973 when Chicago Board Options Exchange [2,p6] started to issue and trade them. In a financial market, options can be bought or sold for a given price. The pricing of options is very important and can lead to disastrous consequences if not done correctly. There are several pricing models such as the Black-Scholes[1], Monte Carlo or binomial. In this essay we will be working with the binomial options pricing model, which is a discrete time model. Let us now look at some financial terms we need in order to understand options.

*Assets* are economic resources such as shares, commodities (gold, silver, oil) or bonds. We can buy and sell these assets in financial markets at any time for a given *market price*, we can do this because we can assume that at a given time there will always be a buyer or a seller ready for the market price. We also assume that a single asset can be subdivided into smaller parts.

In the corporate world, every company needs money to fund various activities such as production or advertising. There are many ways to get this money, however we will focus on *shares*. As an investor you can buy shares of a company for a given price. In doing this you become a part owner of the company. You can buy and sell these shares in a stock market and the price of a single share is called the *stock price*. We can *short sell* shares, which means we can borrow shares and sell them, but we will eventually have to buy the same amount and give them back. This borrowing of shares does not cost any money.

## 1.1 Options

An *option* is a contract which gives a person the right, but not the obligation, to buy or sell an asset for a determined price, known as the *strike price*, when the option expires. A *call* option gives the right to buy the underlying asset and a *put* option gives the right to sell the underlying asset. The underlying asset we will focus on is shares.

[1]Myron Scholes and Robert Merton received Nobel Prize of Economics in year 1997 for their work on the Black-Scholes model. Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish academy.

**Example 1.1.** Let us assume it is January and there exists a company MathsCorp with the stock price of £10 and you think that the stock price is going to be £11 in February. You can then buy a February call option which would allow you to buy 100 shares of MathsCorp for a certain price, say £10 on the $1^{st}$ of February (time of expiry of this option). In doing this you can *exercise* the option and buy 100 shares for £10 pence then sell them at the stock price which we predicted to be £11, making a profit of £100.

**Remark.** The option holder does not have to exercise the option, this means if the investor is going to make a loss, he/she does not have to buy or sell the underlying asset, however the investor will still make a loss due to the price of the option. These options can be traded any time before their expiry date.

Mathematically these terms are defined as follows:

- $t$ is time, where $t \in 0 \cup \mathbb{N}$ as this will be a discrete time model and $t = 0$ is the initial time
- $T$ is the time of expiration of the option
- $S_t$ is the stock price at time $t$
- $K$ is the strike price of the option

**Definition 1.2.** *Payoff* is the amount of money made by the option holder, this does not include the option price. The payoff of an option could be thought of the value of the option. For the call option holder the payoff is:

$$max(S_T - K, 0)$$

and for the put option holder the payoff is:

$$max(K - S_T, 0).$$

## 1.2 Bonds

A *bond* is a debt obligation issued by a corporation or government that includes a promise to repay the principal and interest at a fixed rate. It is an asset that we can buy or sell at any time on the financial markets. After buying a bond, we receive specified periodic payments which are called *interest payments* until we sell the bond. We will assume that every interest payment is immediately invested back into the same bond for the same initial price hence increasing the value of the bonds we hold[2]. The value of bonds increases after a time period $\delta t$ (due to interest payment) to $Be^{c\delta t}$ [3] where $B$ is the value of the bonds at the start of the time period and $c$ some constant with $c \geq 0$, so the interest payment we receive are $B(e^{c\delta t} - 1)$ after the time period.

- $B_t$ is the value of the bond at time $t$

## 1.3 Time Value of Money

We assume that whenever we have money that is not invested into shares the *time value* of that money increases as time increases at a certain rate which we will call the *risk free rate*. We will also assume that we can borrow money at any time where the periodic interest payment we have to pay back is at the same risk

[2]We will see in the next section why we do this.

[3]For more information on bonds look at zero-coupon bonds on [2,p78].

free rate as before until we pay back fully what we borrowed.

A different approach to understanding the above is to assume that as $t$ increases, the value of money also increases with the risk free rate. Let $r \geq 0$ be the rate, which is continuously compounded interest[4], then at the time $t$ from now, one pound will be worth $e^{rt}$ pounds [3,p5]. In mathematical terms let $P$ be a function which tells us how much $P_0$ at time 0 is worth at time $t$ using the following equation:

$$\frac{dP}{P} = rdt,$$

where $r \geq 0$, by solving this ordinary differential equations we get the following for time $t$:

$$P = P_0 e^{rt}$$

- $r \geq$ is the risk free rate

### 1.4 Arbitrage

*Arbitrage* strategy is when someone takes advantage of two or more mispriced asset and making risk free profit. *Risk free* profit means a person is gaining money without risking any of his/her own money. In terms of probability, it is when a person has zero money and from borrowing money at the risk free rate still has a zero probability of losing money and a positive probability of making money. We will assume that there is no arbitrage strategies in this model.

**Example 1.3.** If a share has the stock price of £1 in the UK and that same share has the stock price of £1.20 in the US then a person can take advantage of this by buying the shares in the UK and selling them in the US making a risk free profit with zero probability of losing money.

### 1.5 Portfolio

A *portfolio* is a collection of shares and bonds where the value of the portfolio is the stock price multiplied by the number of number shares plus the bond value multiplied by the number of bonds. We will look further into portfolios in the next section.

- $\phi_t$ is the number of shares in the portfolio at time $t$
- $\psi_t$ is the number of bonds in the portfolio at time $t$
- $V_t = \phi_t S_t + \psi_t B_t$ is the value of the portfolio at time $t$

## 2 Binomial Model

We now want to make our model and to do so we need two main components. First our stock price should have some kind of randomness and second, we want something that follows the time value of money. Each component will have its own model, one for the stock price and another for the time value of money. The big assumption that we make in our model is that arbitrage is not possible. For the stock price we will consider two possibilities that after a time step either the stock price goes up or down. This idea is known as the binomial model. For the time value of money, we will look at bonds. Let us briefly look at some graph theory as it will be useful in these models.

[4]For information about how to derive the continuously compounded interest or see [2,pp77-78].

**Definition 2.1.** A *graph* is a pair $(V, E)$ where $V$ is the set of vertices and $E$ is the set of edges or also thought about as pairs of vertices.

**Definition 2.2.** A *path* is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that no vertex is repeated and $E = \{(v_{i-1}, v_i) : i = 1, 2, \ldots, k\}$[5]. A path graph is a graph which itself is a path.

**Definition 2.3.** We want to define a *tree* [5,p137], for this we will need to know a few more terms. A *parent* vertex $u$ has a *child* which is a vertex $v$ if $(u, v) \in E$[6]. A *root* is a vertex that has no parents. A tree is a graph contains one root, for which there exists a unique path from the root to any vertex in the graph.

**Definition 2.4.** A *full binary* tree is a graph where every vertex has either 2 children or 0 children.

## 2.1 One Step Binomial Model

For modelling the time value of money, we can use a bonds model. When we have money that is not invested in shares we can invest it into bonds where the rate of increase of the value of the bonds is the risk free rate which gives us the same increase as the time value of money. If we want to borrow money we can assume that we are buying 'minus' bonds so that the interest payments we have to pay back match the bonds 'minus' interest payments. We can also see that a path graph would fit these assumptions, as the value of the bond only has one outcome after a time period. The following graph (Figure 1) illustrates our model for time value of money using bonds. At time $t = 0$ the value of the bonds is $B_0$ and after a time period $\delta t$ we can see that it has increased to $B_0 e^{r\delta t}$.

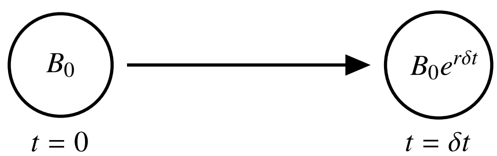



Figure 1: Bond Model

For our binomial model for the stock price we can see how a full binary tree would fit our assumptions where vertices would denote the stock prices and edges show the possible outcomes to the next stock prices. The following graph (Figure 2) illustrates our model where we assume that at time $t = 0$ the stock price is $S_0$ then after a time step $\delta t$ we will assume that the stock price can go up to $uS_0$ with the probability $p$ or go down to $dS_0$ with probability $1 - p$ where $d, u \in \mathbb{R}_{>0}$ and $d < u$.

[5] The order here is important as it gives us a sense of direction where for $(u, v)$ we are going from the vertex $u$ to $v$.

[6] The order here is also important, if $(v, u) \in E$ then $v$ would be the parent vertex and $u$ child, the direction is from the parent vertex to the child vertex.

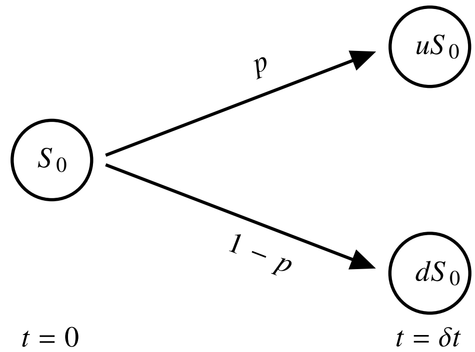



Figure 2: Stock Price Model

We can now state a proposition about $u$ and $d$ from the bond model which is similar to the lemma on [3,p8].

**Proposition 2.5.** For a single time step $\delta t$ we get that $d < e^{r\delta t} < u$.

*Proof.* Let us assume that $d \geq e^{r\delta t}$ then at time $t = 0$ we can borrow enough money to buy one share at $S_0$. Then at time $t = \delta t$ if the stock goes down to $dS_0$ we still don't make a loss when we pay back the money we borrowed as $dS_0 \geq S_0 e^{r\delta t}$ where $S_0 e^{r\delta t}$ is the money we owe. If the stock goes up we know $uS_0 > dS_0 \geq S_0 e^{r\delta t}$ making a profit hence we have a zero probability of making a loss and a positive probability of making profit. This is arbitrage trading hence a contradiction so $d < e^{r\delta t}$.

Assume $u \leq e^{r\delta t}$ then at time $t = 0$ we can short sell a share for $S_0$ and buy bonds with it. So at time $t = \delta t$ we have $S_0 e^{r\delta t}$ by selling those bonds. If the stock price went up to $uS_0$ we don't make a loss as $S_0 e^{r\delta t} \geq uS_0$ and if the stock goes down then we definitely make a profit. This is a contradiction as above due to arbitrage trading. So $e^{r\delta t} < u$. □

Using the stock model we can make a model for an option.

**Example 2.6.** Let us assume that we are looking at a call option model, then using the payoff $(max(S_T - K, 0))$, we know the value of the option at time $T$. Let $K = 110$. The first graph is the stock price model and the graph after is the model for a call option.

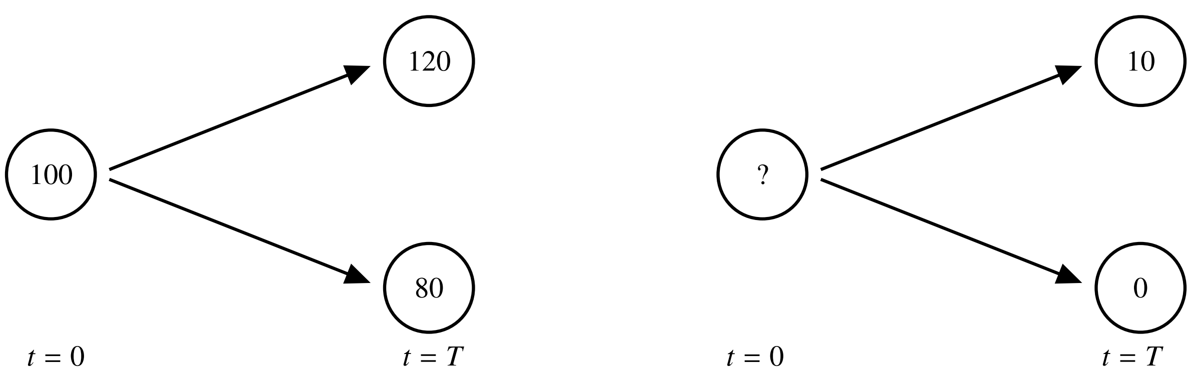



Figure 3: Payoff Model

Now to model the option's price, we can make a portfolio containing a certain numbers of shares and bonds. We know we have to match the value of the portfolio to the value of the option at $t = T$. We can show that if the value of the portfolio at time $T$ matches with the payoff of an option then they must have the same value at time $t = 0$.

**Proposition 2.7.** Any two portfolios which have the same value $V_T$ at time $t = T$ have the same value at $t = 0$. [4,p4]

*Proof.* Assume there are two portfolios $A$ and $C$ with $V_T$ at $t = T$ and with values $V_0$ and $V_0'$ at $t = 0$. Let $V_0 \neq V_0'$ then without loss of generality let us assume $V_0' > V_0$. Then at time $t = 0$ we can short sell portfolio $C$. In doing this we are short selling the shares contained in the portfolio and borrowing the amount that the bonds are worth in the portfolio. At the same time we can buy portfolio $A$ therefore we would now have $V_0' - V_0$ worth of money. Then at $t = T$ we can sell the portfolio $A$ and use this to buy portfolio $C$ which cancel each other out as they have the same value $V_T$ leaving us with $V_0' - V_0$ when we started with zero money. This is a contradiction to arbitrage. Hence $V_0 = V_0'$. $\square$

**Remark.** Using the same proof as above but replacing portfolio $C$ with options we can obtain the same result.

We can now look at how we can use portfolios to model options. Let us consider a portfolio which contains $\phi$ number of shares of a stock and $\psi$ number of bonds. Then our portfolio at time 0 is worth $\phi S_0 + \psi B_0$. Finally at time $T$ it is either worth $\phi u S_0 + \psi B_0 e^{rT}$ or $\phi d S_0 + \psi B_0 e^{rT}$. Now if we look at the following options model where $f_2$ and $f_3$ are derived using the payoff of an option and $f_1$ is unknown. We assume that the value of the portfolio is the same as the payoffs of the option.

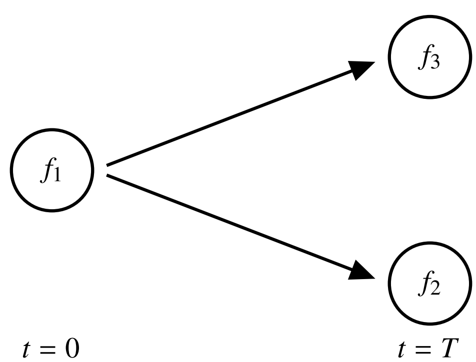



Figure 4: Options Model

From this we know:

$$f_3 = \phi u S_0 + \psi B_0 e^{rT}$$
$$f_2 = \phi d S_0 + \psi B_0 e^{rT}$$

From these equations we can derive:

$$\phi = \frac{f_3 - f_2}{S_0(u - d)}$$
$$\psi = B_0{}^{-1} e^{-rT} \left( f_3 - \frac{u(f_3 - f_2)}{u - d} \right)$$

We have now constructed a portfolio to model the value of an option and so we can now calculate $f_1$.

**Theorem 2.8.**

$$f_1 = \phi S_0 + \psi B_0 = \left(\frac{f_3 - f_2}{u - d}\right) + e^{-rT}\left(f_3 - \frac{u(f_3 - f_2)}{u - d}\right)$$

*Proof.* Assume without loss of generality that $f_1 < \phi S_0 + \psi B_0$. We can then short sell the portfolio and buy the option at $t = 0$ then sell the option at $t = T$ and buy back the portfolio as the payoffs match the values of the portfolio. This leaves us with a $(\phi S_0 + \psi B_0 - f_1)$ gain. This is arbitrage trading, hence a contradiction. □

**Remark.** We can see that the value of the option is not dependent on the probability $p$ of the change of a stock price.

**Example 2.9.** Let us assume the following stock price model shown below on the left, $r = 0$ and $B_0 = 1$. Consider a call option with the strike price $K = 3$ then the graph below (Figure 5) on the right shows the pay offs:

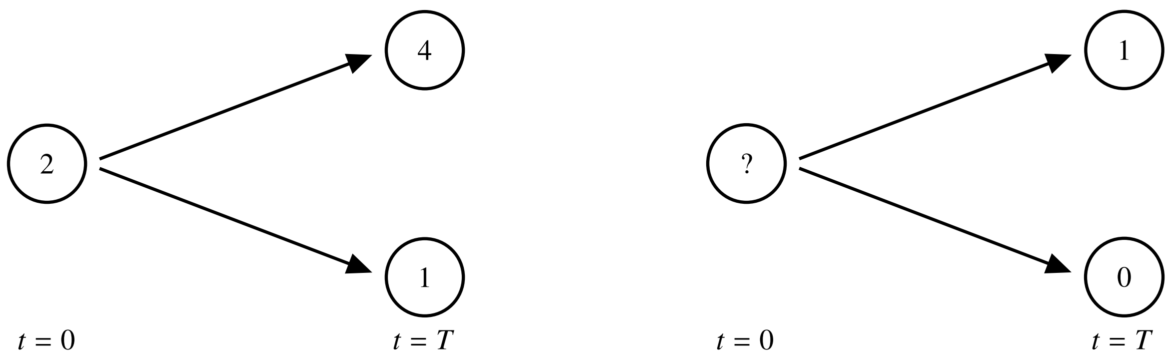



Figure 5: Example of using payoff

From this we obtain that $u = 2$, $d = 0.5$, $f_2 = 0$ and $f_3 = 1$. Using what we derived previously we get that $\phi = \frac{1}{3}$ and $\psi = -\frac{1}{3}$. Hence the price of the option is $f_1 = \phi S_0 + \psi B_0 = \frac{1}{3} \times 2 - \frac{1}{3} \times 1 = \frac{1}{3}$. Briefly looking at how the portfolio matches the option, if the stock price goes up then the option is worth 1 and the portfolio is worth $\phi u S_0 + \psi B_0 = \frac{1}{3} \times 2 \times 2 - \frac{1}{3} \times 1 = 1$. If the stock price goes down then the option is worth zero and the portfolio is worth $\phi d S_0 + \psi B_0 = \frac{1}{3} \times \frac{1}{2} \times 2 - \frac{1}{3} \times 1 = 0$. Remember buying 'minus' bonds is the same as borrowing money which is possible by assumption.

From rearranging the equation for $f_1$ we can write:

$$\begin{aligned} f_1 &= S\left(\frac{f_3 - f_2}{u - d}\right) + e^{-rT}\left(f_3 - \frac{u(f_3 - f_2)}{u - d}\right) \\ &= e^{-rT}\left[\left(1 - \frac{e^{rT} - d}{u - d}\right) f_2 + \left(\frac{e^{rT} - d}{u - d}\right) f_3\right] \end{aligned}$$

We define $q$ as:

$$q := \frac{e^{rT} - d}{u - d}$$

Then:

$$f_1 = e^{-rT}\left[(1 - q) f_2 + q f_3\right]$$

**Corollary 2.10.** $q \in [0, 1]$

*Proof.* This statement follows from the proposition that stated $0 < d < e^{r\delta t} < u$ but for us $\delta t = T$. Assume $q < 0$ then we get that $e^{rT} < d$ which is a contradiction. Similarly assume $q > 1$, then we get $u < e^{rT}$ which is a contradiction. □

## 2.2 Multi Step Binomial Model

We can now extend this tree of one step to an $n$-binomial tree. The bond model is the same but just with more steps. We will assume that we will have $u_i$ and $d_i$ for each time step $t = i$ which will have the same properties as before. Here is an example (Figure 6) of a 2 step model for stock price:

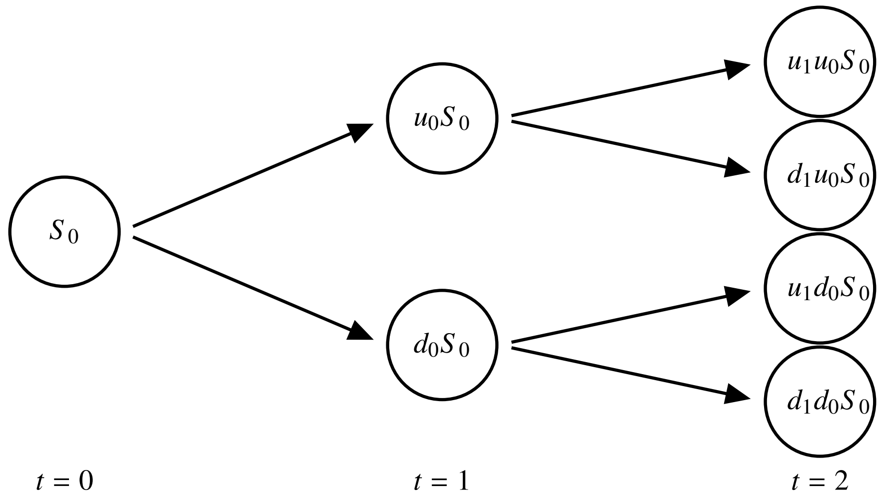



Figure 6: Two Step Stock Price Model

In this model we assume that any vertex before time $t = T$ has two possible values it can go to in the next time step. If the current vertex has the stock price $S$ then it has a probability of $p_i$ to go to $u_iS$ and probability of $1 - p_i$ to go to $d_iS$ in the next time step as shown below (Figure 7).

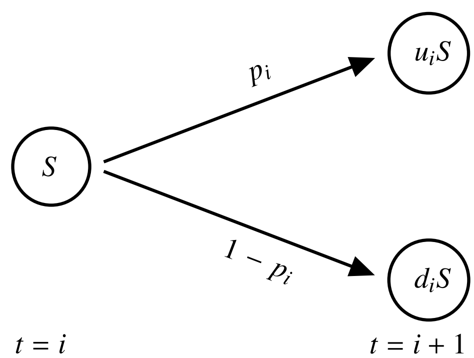



Figure 7: Generalised Multi Step Stock Price Model

With the above 2 step model (Figure 8) and the general model, we can model an options value like we did with a one step model:

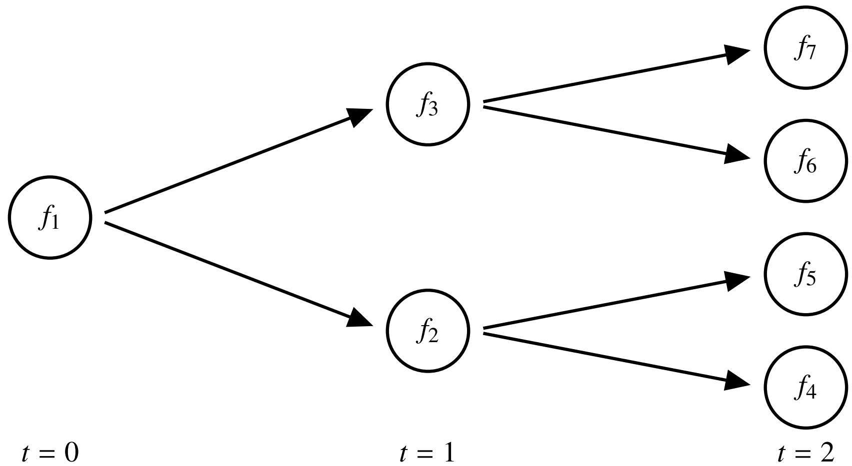



Figure 8: Two Step Options Model

And the generalized model for options (Figure 9):

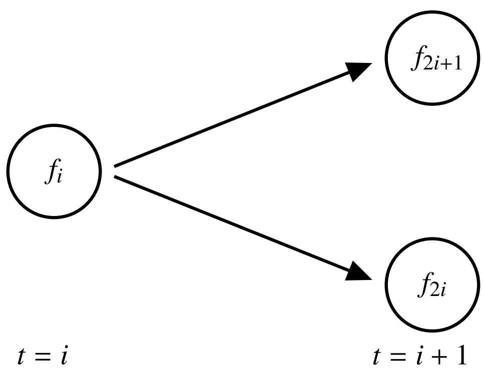



Figure 9: Generalised Options Model

From the one step model we can use induction and still work backwards to find the value of the option at $t = 0$, using the above model the following result is derived:

$$f_i = e^{-ri}\left[(1 - q_i)\, f_{2i} + q_i f_{2i+1}\right]$$

Where $q$ is the same as defined before except it is dependent on the difference between the time steps:

$$q_i := \frac{e^{ri} - d_i}{u_i - d_i}$$

We also extend our portfolio to $(\phi_i, \psi_i)$ which we can use to model our option hence giving us a unique answer.

**Example 2.11.** Suppose we have a call option for the following stock price model (Figure 10) with $K = 100$, $B_0 = 1$ and $r = 0$.

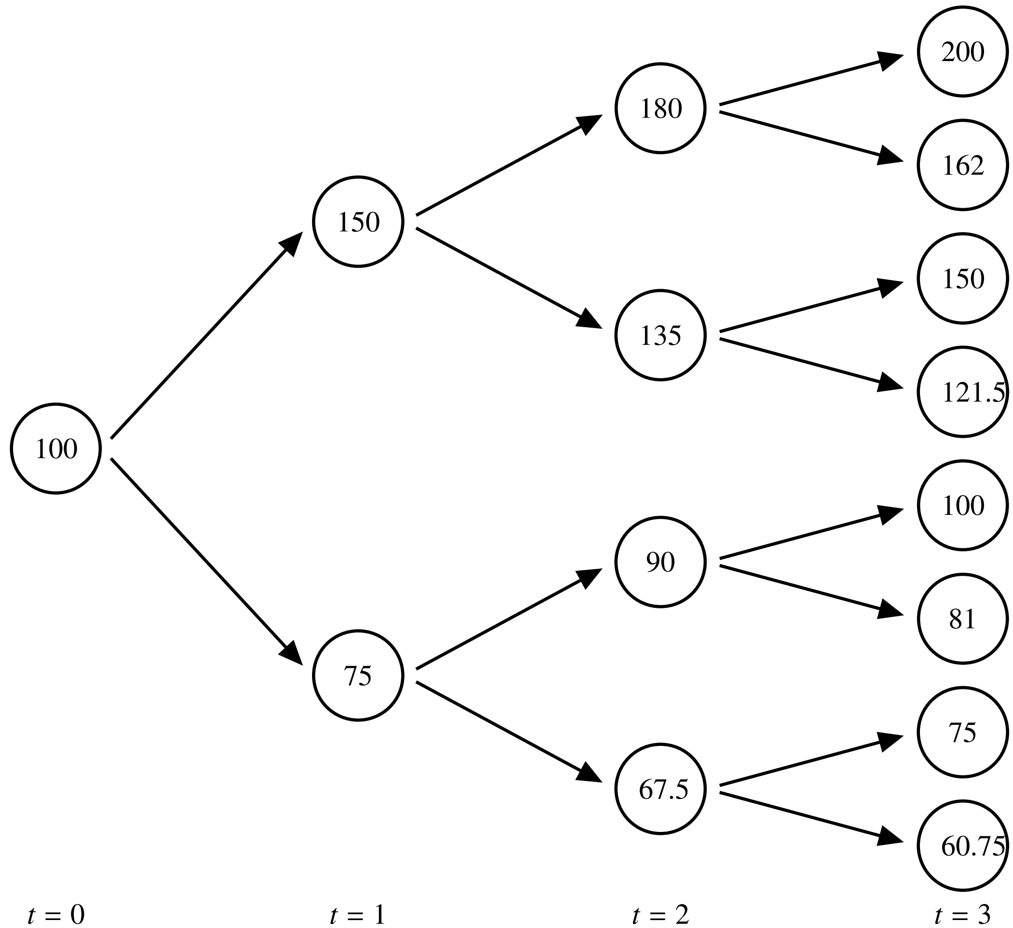



Figure 10: Example of a three step stock price model

From the above graph we obtain that:

| Time $i$ | $u_i$ | $d_i$ |
|---|---|---|
| 0 | 1.5 | 0.75 |
| 1 | 1.2 | 0.9 |
| 2 | $1.\overline{11}$ | 0.9 |
| 3 | - | - |

We can calculate the options model (Figure 11) using the pay off of the call option and the formula we derived for $f_i$ earlier:

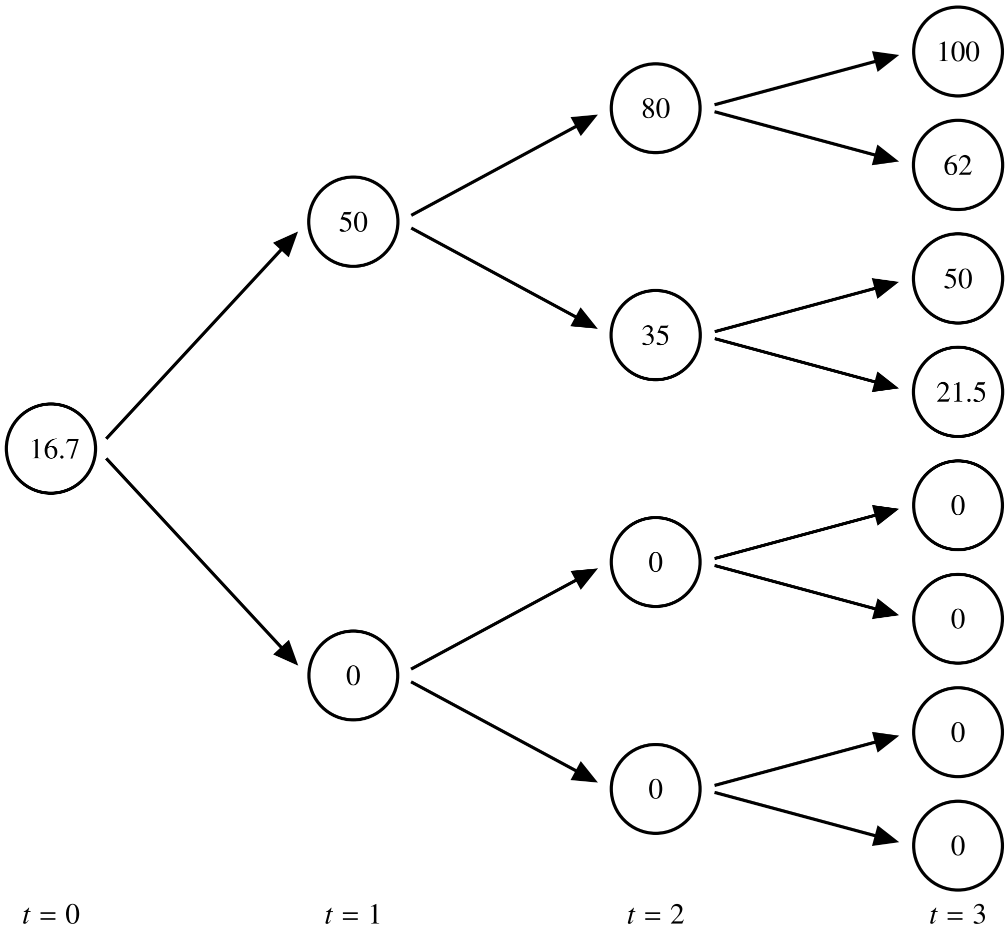



Figure 11: Example of a three step options model

From this we can calculate $\phi$ and $\psi$ at every time $t$ from making assumptions of the stock price going up or down.

- Time 0 The option is worth 16.7 so we calculate $\phi$ as $(50-0)/(100(1.5-0.75))$ which is $\frac{2}{3}$ so we need to borrow 50 to match our portfolio to the option.

  **Suppose stock price goes up to 150**

- Time 1 Now $\phi$ is $(80-35)/(150(1.2-0.9))$ which is 1 unit of the share giving us $1 \times 150$ worth hence we need to borrow 100.

  **Suppose stock price goes up to 180**

- Time 2 Now $\phi$ is $(100-62)/(180(1.\overline{11}-0.9))$ which is 1 unit of the share giving us $1 \times 180$ worth hence we need to borrow 100.

  **Suppose stock price goes down to 162**

- Time 3 The option is now worth 62 and our portfolio is worth $1 \times 162$ minus what we borrowed which is 100 giving us 62.

The table below shows how our portfolio changes. For example $\phi_1$ shows the units of shares held between time $t = 0$ and $t = 1$. The option value will match with both the old and the new portfolios, for instance $F_1 = \phi_1 S_1 + \psi_1 = \phi_2 S_1 + \psi_2$.

| Time $i$ | Last Jump | Stock Price $S_i$ | Option Value $F_i$ | Stock Holding $\phi_i$ | Bond Holding $\psi_i$ |
|---|---|---|---|---|---|
| 0 | - | 100 | 16.7 | - | - |
| 1 | up | 150 | 50 | $0.\overline{66}$ | -50 |
| 2 | up | 180 | 80 | 1.00 | -100 |
| 3 | down | 162 | 62 | 1.00 | -100 |

From this we can see that the multi step binomial model works just as well as the one step model and gives us a unique model for an option and gives us a unique portfolio for each time step which we can work out using the formulas derived.

# 3 Binomial Representation Theorem

Until now we can constructed a model for the option using portfolios. Let us now prove a very important theorem that tells us about the existence of these portfolios. Let us start with some definitions.

**Definition 3.1.** A *probability measure* is a function $\mathbb{P} : \mathcal{P}(\Omega) \to [0, 1]$ where $\Omega$ is a finite sample space and $\mathcal{P}(\Omega)$ is the power set of the sample space which is the set of all subsets of $\Omega$ and $\mathbb{P}$ has the following properties:

- $\mathbb{P}(\Omega) = 1$
- If $A, B \subset \Omega$ such that $A \cap B = \emptyset$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

**Definition 3.2.** A *random variable* is a function from a sample space $\Omega$ to $\mathbb{R}$, associating each possible outcome with a real number.

**Definition 3.3.** We will define a *process* $Z$ to be a tree graph where the vertices denote numerical values. $Z_i$ is a random variable which denotes the value of the process at time $t = i$. We will use $S$ to denote a process which represents the stock price.

**Example 3.4.** The following graph (Figure 12) is the process $S$ of a stock price with probability measure $\mathbb{P}$ where the vertices denote stock prices. This process is also a full binary tree. The probability $\mathbb{P}$ of the stock price going up or down from any vertex is exactly 0.5. $S_1$ can be 90 or 110 depending on what vertex we are on (the outcome).

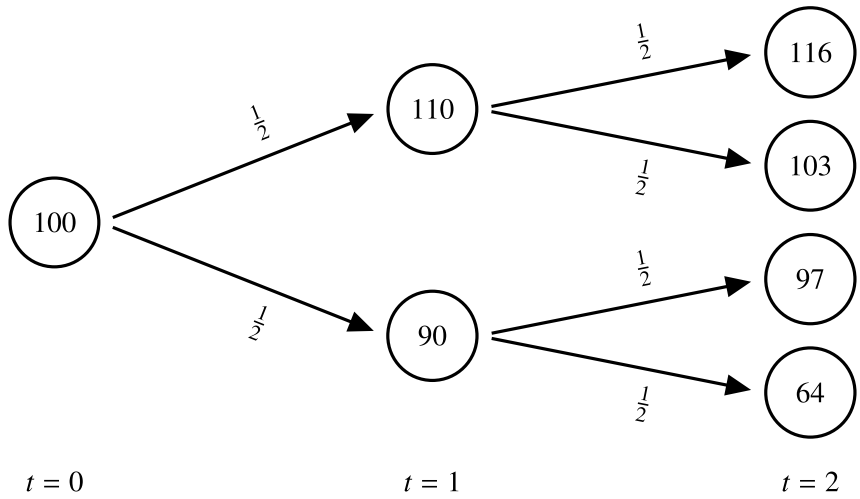



Figure 12: Stock price process

**Example 3.5.** Another example of a process is a simple full binary tree with numbered vertices shown in the following graph (Figure 13). We will use these numbers to denote corresponding vertices in the previous process $S$ (Figure 12).

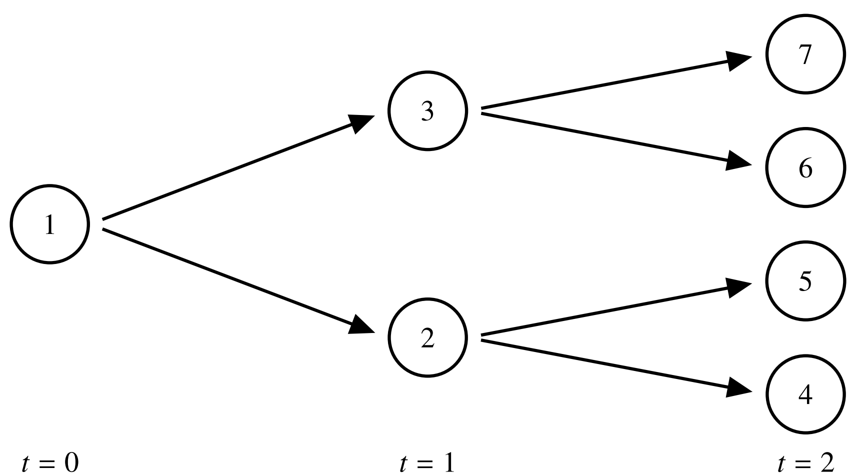



Figure 13: Process with numbered vertices

**Definition 3.6.** A *filtration* $(\mathcal{F}_i)$ is the path in a process up to a vertex at time $t = i$.

**Remark.** The filtration has a one to one correspondence with each vertex as the binary tree structure ensures it. If we choose a vertex then it has a unique path leading up to it. So now when we want to talk about a fixed vertex then we do so by using its filtration or if we want to talk about the set of vertices at time $t = i$ we can denote that by $\{\mathcal{F}_i\}$.

**Example 3.7.** Using the process with numbered vertices illustrated in Figure 13, we can form a filtration table for all the vertices:

| Vertex | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Filtration | (1) | $(1,2)$ | $(1,3)$ | $(1,2,4)$ | $(1,2,5)$ | $(1,3,6)$ | $(1,3,7)$ |

**Definition 3.8.** The *claim* is the image of a function on the set of vertices of a process at time $t = T$ or the image of $f$ where $f : \{\mathcal{F}_T\} \rightarrow \{\mathcal{F}_T\}$, where $\{\mathcal{F}_T\}$ is the set of vertices of a process at time $t = T$. We will denote the claim on a process $S$ where the claim function gives us the payoffs at time $t = T$ by $X$.

**Example 3.9.** From the first example of a process $S$ (Figure 12) if we assume $T = 2$ then $S_2$ itself is a claim which is a result of the identity function and the value of a call option with a strike price $K = 100$ is also a claim.

| Vertex | Claim | |
|---|---|---|
| | $S_2$ | $X = max(S_2 - K, 0)$ |
| 4 | 64 | 0 |
| 5 | 97 | 0 |
| 6 | 103 | 3 |
| 7 | 116 | 116 |

**Remark.** The difference between a claim and a process is that a process is defined for all time $t$ whereas a claim is only defined for time $t = T$.

**Definition 3.10.** Let $X$ be a random variable taking the values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ respectively then the *expected value* of $X$ is:

$$\mathbb{E}(X) = \sum_{i=1}^{n} x_i p_i$$

**Example 3.11.** We will look at the expectation of a one step binary tree (Figure 14):

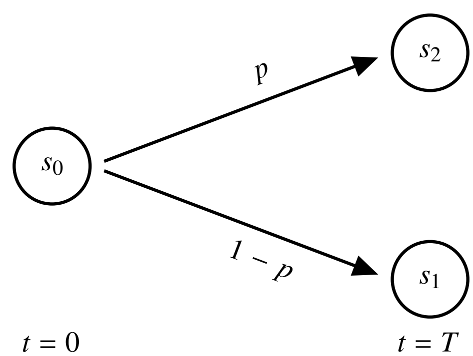



Figure 14: Expectation of a one step binary tree

The expectation of the claim X of this tree where X has the values $s_1$ and $s_2$ with probabilities shown above (where $\mathbb{P}$ is the probability measure) is:

$$\mathbb{E}_{\mathbb{P}}(X) = (1 - p)s_1 + ps_2$$

**Remark.** If we look back at the single and multi step binomial models of the option then what we will find is that we were using the expectation to work backwards to find the value of the option. The probability measure, we were using were the $q$ and $1 - q$ which we defined.

**Properties:**
Let $X$ and $Y$ be random variables in $\Omega$ under the probability measure $\mathbb{P}$ and let $\alpha$ and $\beta$ be any constants then:

- $\mathbb{E}_{\mathbb{P}}(\alpha X + \beta Y) = \alpha \mathbb{E}_{\mathbb{P}}(X) + \beta \mathbb{E}_{\mathbb{P}}(Y)$
- $\mathbb{E}_{\mathbb{P}}(X|Y) = \mathbb{E}_{\mathbb{P}}(X)$

**Definition 3.12.** *Conditional Expectation Operator* $\mathbb{E}_{\mathbb{P}}(\cdot|\mathcal{F}_i)$ [1,p31]
This extends the idea of expectation to two parameters - the probability measure $\mathbb{P}$ and the history $\mathcal{F}_i$. So $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ for a claim $X$ is the expectation of $X$ given the filtration $\mathcal{F}_i$ with respect to $\mathbb{P}$.

**Example 3.13.** Using the first example of a process $S$ (Figure 12), the following shows the conditional expectation operator with filtration(the probability measure $\mathbb{P}$ is exactly 0.5 or going up or down at each split):

| Time $i$ | Filtration Value | Expectation Value $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$ |
|---|---|---|
| 0 | (1) | $\frac{1}{4}(116 + 103 + 97 + 64) = 95$ |
| 1 | (1,3) | $\frac{1}{2}(116 + 103) = 109.5$ |
| | (1,2) | $\frac{1}{2}(97 + 64) = 80.5$ |
| 2 | (1,3,7) | 116 |
| | (1,3,6) | 103 |
| | (1,2,5) | 97 |
| | (1,2,4) | 64 |

From this we can make a process $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$ which looks like (Figure 15), this shows that for any claim $X$ $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ is a process:

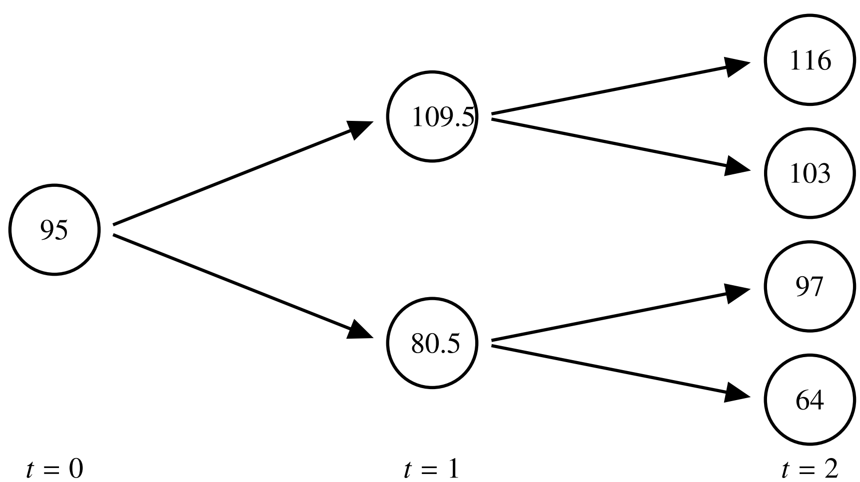



Figure 15: Process of $\mathbb{E}_{\mathbb{P}}(S_2|\mathcal{F}_i)$

**Definition 3.14.** A *previsible* [1,p32] process $\phi = \phi_i$ is a process where the value of a vertex at time $t = i$ only depends on the path up to one time step earlier, $\mathcal{F}_{i-1}$.

**Remark.** This will be important in our portfolio strategy as we would not know if advance where the stock price is going to go.

**Definition 3.15.** A process $S$ is a *martingale* with respect to filtration $(\mathcal{F}_i)$ and a probability measure $\mathbb{Q}$ if:

$$\mathbb{E}_{\mathbb{Q}}(S_j|\mathcal{F}_i) = S_i \quad \forall i \leq j$$

We call this process $S$ a $\mathbb{Q}$ martingale.

**Remark.** This means that the expected value at time $j$ of the process $S$ under probability measure $\mathbb{Q}$ conditional to $\mathcal{F}_i$ is the same as the value at time $i$.

**Lemma 3.16** (Tower Law)**.** If $\forall i \leq j$ we have $\mathcal{F}_i \subset \mathcal{F}_j$ then:

$$\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_j)|\mathcal{F}_i] = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i).$$

**Remark.** The proof of this lemma follows from the second property of the expectation operator.

**Theorem 3.17** (Conditional expectation process of a claim)**.** *For any claim X and a probability measure $\mathbb{P}$, the process $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ is always a $\mathbb{P}$-martingale.*

*Proof.* Let us denote $\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i)$ by $Z_i$ then using the tower law we can show that:

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_j)|\mathcal{F}_i] = \mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_i) \iff \mathbb{E}_{\mathbb{P}}(Z_j|\mathcal{F}_i) = Z_i \quad \forall i \leq j$$

□

**Remark.** To check whether a process $S_i$ is a $\mathbb{P}$-martingale or not we look at its conditional expectation of its terminal value $\mathbb{E}_{\mathbb{P}}(S_T|\mathcal{F}_i)$. If both processes are the same then the process $S_i$ is $\mathbb{P}$-martingale.

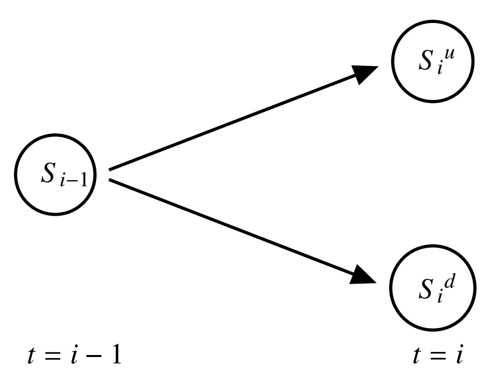
This theorem can be seen in [1,p35] and [3,p44], both of which have similar proofs however the proof in this essay is slightly adapted from both.

**Theorem 3.18** (Binomial Representation Theorem)**.** *Suppose the probability measure $\mathbb{Q}$ is such that the binomial price process $S_i$ is a $\mathbb{Q}$-martingale. If N is any other $\mathbb{Q}$-martingale, then there exists a previsible process $\phi$ such that:*

$$N_i = N_0 + \sum_{k=1}^{i} \phi_k \Delta S_k,$$

*where $\Delta S_i = S_i - S_{i-1}$ is the change in S from time step $i-1$ to i and $\phi_i$ is the value of $\phi$ at the appropriate node at time step i.*

*Proof.* Consider a single branching from a vertex at time $i-1$ to two vertices at time $i$.



Let us denote the change in the processes $S$ and $N$ from time $t = i - 1$ to $t = i$ by:

$$\Delta S_i = S_i - S_{i-1} \quad \text{and} \quad \Delta N_i = N_i - N_{i-1}$$

Given their values at time $(i-1)$, both $S_i$ and $N_i$ can take two values so let us denote them by ${S_i}^u, {S_i}^d$ and ${N_i}^u, {N_i}^d$. We claim that $\Delta N_i = \phi_i \Delta S_i + k_i$ so we need to find $\phi_i$ and $k_i$ such that:

$$ {N_i}^u - N_{i-1} = \phi_i({S_i}^u - S_{i-1}) + k_i$$

and

$$ {N_i}^d - N_{i-1} = \phi_i({S_i}^d - S_{i-1}) + k_i$$

Solving this we get:

$$\phi_i = \frac{{N_i}^u - {N_i}^d}{{S_i}^u - {S_i}^d}$$

and

$$k_i = {N_i}^u - N_{i-1} - \phi_i \left({S_i}^u - S_{i-1}\right). \tag{1}$$

Claim: $\mathbb{E}_{\mathbb{Q}}(\Delta S_i | \mathcal{F}_{i-1}) = \mathbb{E}_{\mathbb{Q}}(\Delta N_i | \mathcal{F}_{i-1}) = 0$

As both $S$ and $N$ are martingales we can show that :

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\Delta S_i | \mathcal{F}_{i-1}) &= \mathbb{E}_{\mathbb{Q}}(S_i - S_{i-1} | \mathcal{F}_{i-1}) \\
&= \mathbb{E}_{\mathbb{Q}}(S_i | \mathcal{F}_{i-1}) - \mathbb{E}_{\mathbb{Q}}(S_{i-1} | \mathcal{F}_{i-1}) \\
&= S_{i-1} - S_{i-1} \\
&= 0
\end{aligned}$$

Similarly we can prove that $\mathbb{E}_{\mathbb{Q}}(\Delta N_i | \mathcal{F}_{i-1}) = 0$. Now if we take the expectation on both sides of the equation (1) then we get that $\mathbb{E}_{\mathbb{Q}}(\phi_i \Delta S_i | \mathcal{F}_{i-1})$ is also zero and it follows that $k_i = 0$. So now we have:

$$\Delta N_i = \phi_i \Delta S_i,$$

then using induction on the increments we get the result we need. □

Now that we have this theorem, if we have a binomial model for our stock price which follows a binary tree process $S$ and if there exists a probability measure $\mathbb{Q}$ such that $S$ is a $\mathbb{Q}$-martingale. Then we could use the theorem to represent another martingale $N_i$ and $\phi$ from the theorem would be part of a portfolio strategy. We have two obstacles before we are finished with the discrete process. First we have a claim $X$ not a martingale, the claim is not a process but just a random variable. The other problem is that we have to take into that we have bonds as well as shares.

Firstly given any probability measure $Q$ we can form the process by taking expectations:

$$E_i = \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_i)$$

If we choose a probability measure $\mathbb{Q}$ such that $S_i$ is a $\mathbb{Q}$-martingale then $E_i$ will also be a $\mathbb{Q}$-martingale due to the binomial representation theorem.

For the bond, if $B_i$ represents time value of money. The process $B_i$ is previsible and positive so without loss of generality let us assume $B_0 = 1$.

- $B_i{}^{-1}$ is also a previsible and positive process.
- $Z_i := B_i{}^{-1} S_i$ is a well defined process.
- $B_T{}^{-1} X$ is also a claim.

So let us assume we have a process $Z$ that is a $\mathbb{Q}$-martingale, then another $\mathbb{Q}$-martingale process $E_i$ can be made by taking expectations of the claim $B_T{}^{-1} X$ such that $E_i = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$. By the binomial representation theorem there exists a previsible process $\phi$ such that:

$$E_i = E_0 + \sum_{k=1}^{i} \phi_k \Delta Z_k,$$

Let us consider the following construction strategy, at time $i$, make the following portfolio $\Pi_i$:

- $\phi_{i+1}$ units of stock $S$,
- $\psi_{i+1} = \left(E_i - \phi_{i+1} B_i{}^{-1} S_i\right)$ units of bond.

Suppose at the time step $i$ we buy this portfolio then it will cost us:

$$\phi_{i+1} S_i + \left(E_i - \phi_{i+1} B_i{}^{-1} S_i\right) B_i = E_i B_i.$$

The value of the above portfolio at time step $i+1$:

$$\begin{aligned} \phi_{i+1} S_{i+1} + \left(E_i - \phi_{i+1} B_i{}^{-1} S_i\right) B_{i+1} &= \left[\phi_{i+1}\left(\frac{S_{i+1}}{B_{i+1}}\right) + E_{i+1}\right] B_{i+1} \\ &= B_{i+1} E_{i+1} \qquad \text{(by binomial representation theorem)} \end{aligned}$$

which is exactly how much the new portfolio $\Pi_{i+1}$ at time $i+1$. At time $T$, the portfolio $\Pi_{T-1}$ is worth $B_T B_T{}^{-1} X$ which is X, the claim we require. There is no arbitrage strategy in this construction as we repeated the argument from the section before when we created a portfolio.

In conclusion there are three steps of pricing an option with a claim $X$:

- Find a probability measure $\mathbb{Q}$ such that the stock price process is a $\mathbb{Q}$-martingale with respect to its filtration.
- Form the process $E_i$ such that
$$E_i = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_i)$$
- Work out the previsible process $\phi$ such that
$$\Delta E_i = \phi_i \Delta Z_i$$

Once we achieve the above steps then the value at time step $i$ of an option, with a claim $X$ at $t = T$. is:

$$\boxed{B_i = \mathbb{E}_{\mathbb{Q}}(B_T{}^{-1} X|\mathcal{F}_i)}$$

So if we have a binary tree process for the stock price and a probability measure $\mathbb{Q}$ such that the process is a $\mathbb{Q}$-martingale then we can make a binary tree process for the value of the option hence giving us the price of the option at any time step because we can always find a portfolio that works. It is not hard to find a process that is a martingale. The model where we defined $q$ where $S_0$ goes to $uS_0$ or $dS_0$ is a martingale by construction.

## References

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